A Regular Heptadecagon is Constructible

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Introduction

For centuries people learned mathematics by studying Euclid's *Elements*. A focus of the *Elements* is on the construction of geometric figures using a straightedge (a ruler with no markings) and a compass. The *Elements* gives constructions of regular polygons with $n = 3, 4, 5, 15$ sides (and polygons with $2^k n$ sides), but not until two thousand years later was the construction of another regular polygon discovered. In 1796 Carl Friedrich Gauss awoke one morning just before his 19th birthday, and by "concentrated thought" discovered that a *regular heptadecagon* (a regular polygon with 17 sides) is constructible.

In Euclidean geometry constructions are given as step-by-step instructions: starting with given points and lines, lines and circles are constructed and their points of intersection are used to construct new lines and circles. Gauss's proof is revolutionary because he did not give any geometric constructions whatsoever. Instead, he defined constructibility as a property of expressions using integers and arithmetical operators, and then using algebra alone proved that the length of a side of a heptadecagon is given by an expression with the property required for constructibility. The algebra required is accessible with a knowledge of secondary-school mathematics.

Constructibility

The first proposition in the *Elements* claims that an equilateral triangle can be constructed. (Henceforth, "construct" will be used as an abbreviation for "construct by straightedge and compass.") Given a line segment $\overline{AB}$, draw two circles whose centers are A and B, and whose radii are the length of $\overline{AB}$. C, the intersection of the circles, defines the third vertex of the triangle (Figure 1).

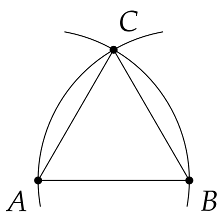

Figure 1: Construction of an equilateral triangle

Definition: A real number x is *constructible* if and only if starting with a line segment defined to be of length 1 it is possible to construct a line segment of length x.

Theorem: x is constructible if and only if it is the result of evaluating an expression that uses only the integer 1 and the operations $\{+, -, \times, /, \sqrt{}\}$.

Examples: Figure 2 (left) shows how to construct $a - b = 3$ and $a + b = 7$ given $a = 5$ and $b = 2$ by constructing a circle of radius 2 whose center is an endpoint of a. Figure 2 (right) shows how to construct $\sqrt{a} = \sqrt{3}$ given 1 and $a = 3$ by constructing similar triangles within a semicircle.

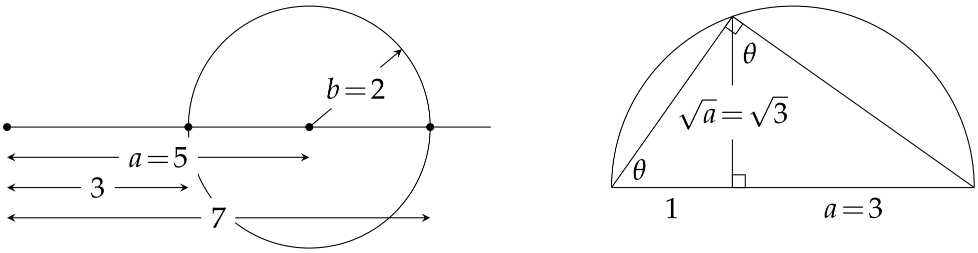

Figure 2: Construction of $a \pm b$, $\sqrt{a}$

These constructions can be generalized to prove that an expression using $\{+, -, \sqrt{}\}$ can be constructed. Expressions using $\{\times, /\}$ can be constructed using similar triangles (Figure 3).

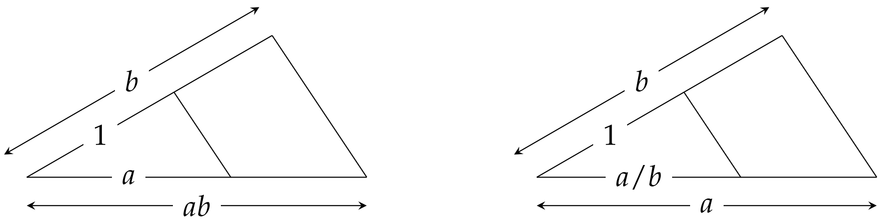

Figure 3: Construction of $ab, a/b$

To prove the forward direction of the theorem, note that lines are defined by linear equations and circles by quadratic equations. By solving pairs of equations, it can be shown that the points of intersection of lines and circles are given by expressions using $\{+, -, \times, /, \sqrt{}\}$, as are the lengths of line segments which are the distances between two points.

Impossible constructions

The Greeks were unable to trisect an angle (divide a given angle into three equal parts), square a circle (construct a square with the same area as a given circle) and double a cube (construct a cube with twice the volume of a given cube). During the nineteenth century it was proved that these constructions are impossible.

Of equal interest was the construction of regular polygons. The Greeks were unable to construct regular polygons with $n = 7, 9, 11, 13, 14, 17, 18, \ldots$ sides, but not until the work of Gauss was any progress made to determine if the construction of these polygons is possible or not.

The mathematics of constructibility

A regular polygon can be inscribed in a unit circle; all of its sides are equal and as are all of its central angles (Figure 4). Clearly, if we can construct a central angle we can construct the side of the polygon that subtends it. Construction has been defined only for line segments, so we first show that given a line segment of length $\cos\theta$, where θ is the central angle of a regular polygon, the polygon is constructible.

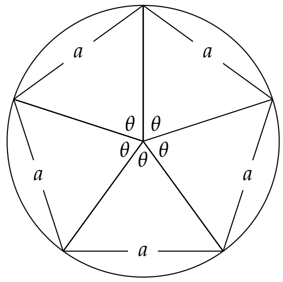

Figure 4: A regular polygon (pentagon) inscribed in a circle

Construct a unit circle centered at O and let A be an arbitrary point on the circle. Construct the radius $\overline{OA}$ and construct the (given) line segment $\overline{OB}$ of length $\cos\theta$ on $\overline{OA}$. Construct the perpendicular at B (Figure 5). By the definition of cosine, C, the intersection of the perpendicular with the circle, defines the central angle θ and thus $\overline{AC}$ is a side of the polygon.

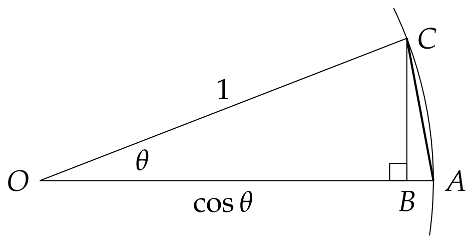

Figure 5: The side of a regular polygon and its central angle

Let us look at some examples:

- The central angle of an equilateral triangle is 120° and $\cos 120^\circ = -1/2$ is constructible, so the equilateral triangle is constructible.
- The central angle of a regular pentagon is 72° and it is not too hard to show that $\cos 72^\circ = (\sqrt{5}-1)/4$, so a regular pentagon is constructible.
- The central angle of regular pentadecagon (a regular polygon with 15 sides) is $360^\circ/15 = 24^\circ = (120^\circ - 72^\circ)/2$. This can be constructed from an equilateral triangle (dashed blue) and regular pentagon (dashed red) with a common vertex and a common radius of a central angle (Figure 6). $\angle AOB$ (red) is the central angle of the pentagon, $\angle AOC$ (blue) is the central angle of the triangle and their difference is $\angle BOC = 48^\circ$, which can be bisected to obtain 24°.

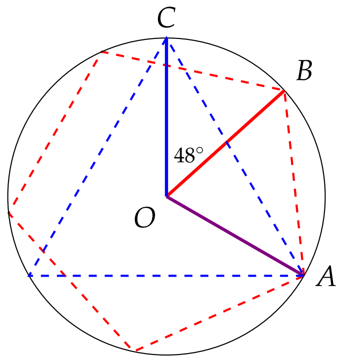

Figure 6: Central angles of an equilateral triangle and a pentagon

Alternatively, trigonometric identities can be used to define $\cos 24^\circ$ from lengths previously shown to be constructible:

$$\begin{aligned}\cos 48^\circ &= \cos(120^\circ - 72^\circ) = \cos 120^\circ \cos 72^\circ + \sin 120^\circ \sin 72^\circ \\ \cos 24^\circ &= \cos \frac{48^\circ}{2} = \sqrt{\frac{1+\cos 48^\circ}{2}}\end{aligned}$$

Gauss's goal was to show that $\cos(360^\circ/17)$ is constructible.

Mathematical prerequisites

With one exception we will only use mathematics usually taught in secondary school:

- Exponents: $x^n \cdot x^m = x^{n+m}$.
- Integer division: given n, d, there exist $q, 0 \leq r < d$ such that $n = qd + r$.
- Quadratic polynomials: the roots of $x^2 + bx + c$ are $\frac{-b\pm\sqrt{b^2-4c}}{2}$.
- Trigonometric identities for $\cos(\alpha \pm \beta), \sin(\alpha \pm \beta)$.
- Complex numbers: points on the unit circle can be represented as $\cos\theta + i \sin\theta$. de Moivre's formula: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i \sin n\theta$.

The exception is the Fundamental Theorem of Algebra, which states that an n degree polynomial with complex coefficients has n complex roots. Actually, we only need one simple case of the theorem: the polynomial $x^n - 1$ has (at least) one root $r \neq 1$ or, even more narrowly, that $x^{17} - 1$ has (at least) one root $r \neq 1$

The roots of unity

The roots of $x^n - 1$ are called the *n-th roots of unity*. If r is an n-th root of unity then $(r^2)^n = (r^n)^2 = 1^2 = 1$ so r^2 is also an n-th root of unity. It follows that $1, r, r^2, \ldots, r^{n-2}, r^{n-1}$ are all n-th roots of unity. What we don't know is if they are *distinct* and thus all the n-th roots of unity.

- The roots of the polynomial $x^4 - 1$ are $\{1, -1, i, -i\}$. By computation we find that $\{i^0, i^1, i^2, i^3\} = \{1, i, -1, -i\}$, but $\{(-1)^0, (-1)^1, (-1)^2, (-1)^3\} = \{1, -1\}$, so the powers of -1 are not distinct.

- The roots of the polynomial $x^3 - 1$ are:

$$x_0 = 1, \quad x_1 = \frac{-1 + i\sqrt{3}}{2}, \quad x_2 \frac{-1 - i\sqrt{3}}{2}.$$

  By computation we find that $\{x_1^0, x_1^1, x_1^2\} = \{x_0, x_1, x_2\}$ and similarly $\{x_2^0, x_2^1, x_2^2\} = \{x_0, x_2, x_1\}$, so the powers of the two roots ($\neq 1$) are distinct.

Theorem: If n is a *prime number* and $r \neq 1$ is an n-the root of unity then $1, r, \ldots, r^{n-1}$ are distinct.

Proof: If $r^i = r^j$ for some $0 \leq i < j \leq n-1$ then $r^j / r^i = r^{j-i} = 1$. Let m be the smallest positive integer such that $r^m = 1$. By the division formula:

$$1 = r^n = r^{ml+k} = (r^m)^l \cdot r^k = 1^l \cdot r^k = r^k,$$

where $0 \leq k < m$. But $0 < k < m$ and $r^k = 1$ contradict the assumption that m was the smallest such positive number, so $k = 0$ and $n = ml$ is not prime.

From roots to coefficients of polynomials

Suppose that we know the values of r_1, r_2, two roots of a quadratic polynomial $x^2 + bx + c$. What are the coefficients? By computation:

$$(x - r_1)(x - r_2) = x^2 - (r_1 + r_2)x + r_1 r_2 = x^2 + bx + c,$$

so:

$$b = -(r_1 + r_2), \qquad c = r_1 r_2. \tag{1}$$

Similarly the coefficients of any polynomial can be computed if the roots are known. Given that the roots of $x^{17} - 1$ are $\{1, r, r^2, \cdots, r^{15}, r^{16}\}$, we can find the coefficients of the polynomial by multiplying $(x-1)(x-r^1)(x-r^2)\cdots(x-r^{15})(x-r^{16})$. The coefficient of x^{16} is:

$$-(1 + r^1 + r^2 + \cdots + r^{15} + r^{16}),$$

which is zero in $x^{17} - 1$ so:

$$r^1 + r^2 + \cdots + r^{15} + r^{16} = -1. \tag{2}$$

Quadratic polynomials from the 17-th roots of unity

Gauss's insight was that we need not work with the roots in their natural order $r, \cdots, r^{16}$, but that other powers of roots can generate all the roots.

Is there a number g such that the sequence $g^0, g^1, g^2, \ldots, g^{16}$ (all modulo 17) gives all the numbers in the sequence $1, 2, \ldots, 16$ in a different order? Well, not all numbers work; for $g = 4$, we get only four distinct numbers:

$$1, \;\; 4, \;\; 16, \;\; 64 = (17 \cdot 3 + 13) = 13, \;\; 256 = (17 \cdot 15) + 1 = 1.$$

However, for $g = 3$, the sequence $g^0, g^1, g^2, \ldots, g^{16}$ gives of all the numbers in the sequence $1, 2, \ldots, 16$ in a different order:

$$1,\ 3,\ 9,\ 10,\ 13,\ 5,\ 15,\ 11,\ 16,\ 14,\ 8,\ 7,\ 4,\ 12,\ 2,\ 6\,.$$

The reason is that 3 is relatively prime to 16 (and is, in fact, the smallest such integer). Using $g^0, g^1, g^2, \ldots, g^{16}$ as exponents we obtain a reordering of the powers of the root r:

$$r^1,\ r^3,\ r^9,\ r^{10},\ r^{13},\ r^5,\ r^{15},\ r^{11},\ r^{16},\ r^{14},\ r^8,\ r^7,\ r^4,\ r^{12},\ r^2,\ r^6\,.$$

Write the sequence of roots as follows in order to distinguish roots in the odd positions from those in the even positions:

$$\begin{array}{cccccccccccccccc} r^1 & & r^9 & & r^{13} & & r^{15} & & r^{16} & & r^8 & & r^4 & & r^2 & \\ & r^3 & & r^{10} & & r^5 & & r^{11} & & r^{14} & & r^7 & & r^{12} & & r^6\,. \end{array}$$

Let a_0, a_1 be the sums of the roots in the odd and even positions, respectively:

$$\begin{aligned} a_0 &= r + r^9 + r^{13} + r^{15} + r^{16} + r^8 + r^4 + r^2 \\ a_1 &= r^3 + r^{10} + r^5 + r^{11} + r^{14} + r^7 + r^{12} + r^6\,. \end{aligned}$$

Compute $a_0 + a_1$ using Equation 2:

$$a_0 + a_1 = r + r^2 + \cdots + r^{16} = -1\,.$$

Now compute $a_0 a_1$ and simplify. It takes quite a lot of work [1, Section 16.4], but the result is the sum of four copies of Equation 2 so:

$$a_0 a_1 = -4\,.$$

Given that a_0, a_1 are roots, by Equation 1 they are the roots of the polynomial:

$$y^2 + y - 4 = 0\,,$$

and their values are:

$$a_0, a_1 = \frac{-1 \pm \sqrt{17}}{2}\,.$$

Let b_0, b_1, b_2, b_3 be the sums of every fourth root starting from r^1, r^3, r^9, r^{10}, respectively:

$$\begin{aligned} b_0 &= r^1 + r^{13} + r^{16} + r^4 \\ b_1 &= r^3 + r^5 + r^{14} + r^{12} \\ b_2 &= r^9 + r^{15} + r^8 + r^2 \\ b_3 &= r^{10} + r^{11} + r^7 + r^6\,. \end{aligned}$$

Check that $b_0 + b_2 = a_0, b_1 + b_3 = a_1$ and compute the corresponding products which are $b_0 b_2 = b_1 b_3 = -1$. Therefore, b_0, b_2 are the solutions of $y^2 - a_0 y - 1 = 0$, and b_1, b_3

are the solutions of $y^2 - a_1 y - 1 = 0$. Using the values previously computed for a_0, a_1 we can compute the roots b_0, b_1. The results are:

$$b_0, b_1 = \frac{(-1 \pm \sqrt{17}) + \sqrt{34 \mp 2\sqrt{17}}}{4}\,.$$

(For b_0 take plus and then minus and for b_1 take minus and then plus.)

Finally, let c_0, c_4 be the sums of every eighth root starting with r^1, r^{13}, respectively:

$$\begin{aligned} c_0 &= r^1 + r^{16} \\ c_4 &= r^{13} + r^4 \\ c_0 + c_4 &= = b_0 \\ c_0 c_4 &= b_1\,. \end{aligned}$$

c_0, c_4 are the roots of $y^2 - b_0 y + b_1 = 0$:

$$c_0, c_4 = \frac{b_0 \pm \sqrt{(-b_0)^2 - 4b_1}}{2}\,,$$

which after a lot of messy algebra reduces to:

$$\begin{aligned} c_0 = &-\frac{1}{8} + \frac{1}{8}\sqrt{17} + \frac{1}{8}\sqrt{34 - 2\sqrt{17}} \; + \\ &\frac{1}{8}\sqrt{68 + 12\sqrt{17} + 2(-1 + \sqrt{17})\sqrt{34 - 2\sqrt{17}} - 16\sqrt{34 + 2\sqrt{17}}}\,. \end{aligned}$$

What do these solutions of quadratic equations whose coefficients are derived from the roots of $x^{17} - 1$ have to do with the heptadecagon?

Connecting the quadratic expression to the heptadecagon

Actually, we know an expression for a root of $x^{17} - 1$, although the expression does not show that the value is constructible:

$$r = \cos\frac{2\pi}{17} + i \sin\frac{2\pi}{17}\,.$$

This follows from de Moivre's formula:

$$r^{17} = \left(\cos\frac{2\pi}{17} + i \sin\frac{2\pi}{17}\right)^{17} = \cos 2\pi + i \sin 2\pi = 1\,.$$

We found a constructible expression for $r^1 + r^{16}$ so let's see where this leads us:

$$r^1 + r^{16} = \left(\cos\frac{2\pi}{17} + \sin\frac{2\pi}{17}\right) + \left(\cos\frac{16 \cdot 2\pi}{17} + \sin\frac{16 \cdot 2\pi}{17}\right) = 2\cos\frac{2\pi}{17}\,,$$

since:

$$\cos\frac{16 \cdot 2\pi}{17} = \cos\left(\frac{34\pi}{17} - \frac{2\pi}{17}\right) = \cos 2\pi \cos\frac{2\pi}{17} - \sin 2\pi \sin\frac{2\pi}{17} = \cos\frac{2\pi}{17}\,,$$

and similarly $\sin(16 \cdot 2\pi/17) = -\sin(2\pi/17)$. It follows that:

$$\frac{c_0}{2} = \frac{r^1 + r^{16}}{2} = \cos\frac{2\pi}{17}\,.$$

Therefore, the cosine of the central angle of a heptadecagon is the result of evaluating an expression that uses only the integer 1 and the operations $\{+, -, \times, /, \sqrt{}\}$, so a regular heptadecagon is constructible using a straightedge and compass!

The formula that usually appears in the literature is:

$$\begin{aligned}\cos\frac{2\pi}{17} &= -\frac{1}{16} + \frac{1}{16}\sqrt{17} + \frac{1}{16}\sqrt{34 - 2\sqrt{17}} \\ &\quad + \frac{1}{8}\sqrt{17 + 3\sqrt{17} - \sqrt{34 - 2\sqrt{17}} - 2\sqrt{34 + 2\sqrt{17}}}\,.\end{aligned}$$

We leave it to the reader to derive this formula from the one we derived above.

Gauss-Wantzel Theorem

Theorem: A regular polygon with n sides is constructible if and only if n is the product of a power of 2 and zero or more *distinct* Fermat numbers $2^{2^k} + 1$ which are prime.

Gauss proved that the condition for constructibility is sufficient and Pierre Laurent Wantzel proved that the condition is necessary.

The known Fermat primes are:

$$F_0 = 3, \quad F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537\,,$$

and it has been proven that $F_5, \ldots, F_{32}$ are not prime.

The equilateral triangle is constructible by $F_0 = 3$, the regular polygon by $F_1 = 5$, the regular heptadecagon by $F_2 = 17$. The Greeks were also able to construct the regular pentadecagon which is constructible by $F_0F_1 = 15$. By the theorem the regular polygons with $3 \cdot 17 = 51$, $5 \cdot 17 = 85$ and $3 \cdot 5 \cdot 17 = 255$ sides are constructible.

A regular polygon with $F_3 = 257$ sides was constructed by Magnus Georg Paucker in 1822 and by Friedrich Julius Richelot 1832. In 1894 Johann Gustav Hermes claimed to have constructed a regular polygon with $F_4 = 65537$ sides.

Conclusion

Until the late eighteenth century mathematical theorems were proved geometrically. Even Newton who invented the calculus used it as a means of discovery always proved theorems using geometry. Gauss proved the constructibility of the heptadecagon without giving a geometric construction. In fact, the first constructions were not published until almost a century later (a modern construction is given in [3]). His algebraic solution led to the growing ascendancy of algebra during the nineteenth century and to the birth of fields of modern mathematics like algebraic geometry and algebraic topology.

Acknowledgment

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References

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